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Properties of Hermitian operator: → There are two important properties of Hermitian operator.

(I) The eigenvalues are real (positive or negative)

(II) Eigenfunctions corresponding to different eigenvalues are orthogonal to each other.

These two important theorem about Hermitian operators can be proved easily as follows —

(I) Eigenvalues of a Hermitian operator are real.

Let $\hat{A}$ be a Hermitian operator
ψ is eigenfunction
λ is eigenvalues

Then, $\hat{A}\psi = \lambda\psi$ —— (1)

& $(\hat{A}\psi)^* = \lambda^*\psi^*$ —— (2)

Multiplying eq^n (1) by ψ^* & (2) by ψ & then integrating.

$$\int \psi^* \hat{A}\psi \, d\tau = \int \psi^* \lambda \psi \, d\tau = \lambda \int \psi^* \psi \, d\tau$$

and $\int \psi (\hat{A}\psi)^* d\tau = \int \psi \lambda^* \psi^* d\tau = \lambda^* \int \psi \psi^* d\tau$

But since $\hat{A}$ is Hermitian,

$$\int \psi^* (\hat{A}\psi) d\tau = \int \psi (\hat{A}\psi)^* d\tau$$

Therefore, $\lambda \int \psi^* \psi \, d\tau = \lambda^* \int \psi \psi^* d\tau$

or $\lambda = \lambda^*$ i.e. λ is real —— (3)

(II) Eigenfunctions of a Hermitian operator corresponding to different eigenvalues are orthogonal

Let, ψ_1 and ψ_2 = Two eigenfunctions
λ_1 and λ_2 = " eigenvalues respectively of a Hermitian operator $\hat{A}$

The condition of orthogonality is —

$$\int \psi_1 \psi_2 \, d\tau = 0 \text{ or } \int \psi_1 \psi_2^* d\tau = 0, \int \psi_1^* \psi_2 \, d\tau = 0$$

The eigen value eq^ns are, $\hat{A}\psi_1 = \lambda_1 \psi_1$ —— (1), $\hat{A}\psi_2 = \lambda_2 \psi_2$ —— (2)

Now, multiplying eq^n (1) by ψ_2^* and integrating,

$$\int \psi_2^* \hat{A} \psi_1 \, d\tau = \int \psi_2^* \lambda_1 \psi_1 \, d\tau = \lambda_1 \int \psi_2^* \psi_1 \, d\tau$$

But, since $\hat{A}$ is Hermitian,

$$\int \psi_2^* \hat{A} \psi_1 \, d\tau = \int \psi_1 (\hat{A}\psi_2)^* d\tau = \int \psi_1 (\lambda_2 \psi_2)^* d\tau$$

$= \lambda_2^* \int \psi_1 \psi_2^* d\tau$

(ii) $= \lambda_2 \int \psi_1 \psi_2^* d\tau$

(ii) Thus, $\lambda_1 \int \psi_2^* \psi_1 \, d\tau = \lambda_2 \int \psi_1 \psi_2^* d\tau$ or $(\lambda_1 - \lambda_2) \int \psi_1 \psi_2^* d\tau = 0$

But, $\lambda_1 \neq \lambda_2$, $\int \psi_1 \psi_2^* d\tau = 0$

If However $\lambda_1 = \lambda_2$ (i.e. ψ_1, ψ_2 are degenerate) then integral, $\int \psi_1 \psi_2^* d\tau$ need not be zero, Hence ψ_1 & ψ_2 may not be orthogonal